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Equality of two strongly unique minimal projection constants

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Abstract

Let $\mathcal{P}(X, Y)$ denote the set of all linear, continuous projections from a Banach space X onto a linear subspace Y. Let $\hat{f} = (f_1, \dots, f_k) \in \mathbb{R}^k$ be such that $0 < f_1 \leqslant f_2 \leqslant \cdots \leqslant f_k$, $\sum_{i=1}^{k} f_i = 1$. Define $f^{(0)} = (f_1, f_2, \dots, f_k, 0, \dots, 0)$, $f^{(j)} = (0, \dots, 0, 1_{j+k}, 0, \dots, 0)$ for $j = 1, 2, \dots, n-k$. Let $\hat{H} = \ker \hat{f}$ and $H = \bigcap_{j=0}^{n-k} \ker f^{(j)}$. In this paper we prove that the strongly unique minimal projection constant (SUP-constant) of the space $\mathcal{P}(l_\infty^{(k)}, \hat{H})$ is equal to the SUP-constant of the space $\mathcal{P}(l_\infty^{(n)}, H)$. This solves a conjecture stated in Odyniec and Prophet (2007) [12, p. 120]. The main tool applied in our proof is a Kolmogorov type theorem for the strongly unique best approximation.

Keywords: Minimal projection; Strong unicity; Strongly unique minimal projection constant; SUP-constant

1. Introduction

In this paper we discuss equality of two strong unicity constants in the minimal projection problem. The notion of strong unicity was introduced by Newman and Shapiro [10]. Let us recall it now.

Let X be a normed space and let $Y \subset X$ be a nonempty subset. An element $y \in Y$ is called a strongly unique best approximation (for short SUBA) to $x \in X$ if and only if there exists $r > 0$

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such that for every $v \in Y$

$$\|x - v\| \geqslant \|x - y\| + r\|v - y\|. \tag{1}$$

The biggest constant r satisfying (1) is called the strong unicity constant.

The significance of this notion can be illustrated by its two main applications. The error estimate of the Remez algorithm is based on an iteration process for finding the constant r satisfying (1). The strong unicity of the best approximation yields the Lipschitz continuity of the best approximation mapping (see e.g. [4, p. 82]).

Now denote by $\mathcal{L}(X, Y)$ the set of all linear, continuous operators from X into Y and by $\mathcal{P}(X, Y)$ the set of all projections going from X onto Y.

In the case of projections, the notion of strong unicity reduces to the following definition:

Definition 1. Let $P_0 \in \mathcal{P}(X, Y)$. Then P_0 is called a strongly unique minimal projection (we will write this as a SUM-projection for brevity) if and only if there exists $r > 0$ such that for any $P \in \mathcal{P}(X, Y)$

$$\|P\| \geqslant \|P_0\| + r\|P - P_0\|. \tag{2}$$

The largest possible constant for which the inequality in (2) holds is called a strongly unique projection constant (for short, a SUP-constant).

Note that any SUM-projection is the unique minimal projection in $\mathcal{P}(X, Y)$.

Now let us introduce some notation. By S_X we denote the unit sphere in a normed space X and by $\operatorname{ext} S_X$ the set of its extreme points. Let X be a Banach space and let $Y \subset X$ be its closed subspace. Set

$$E(x) = \{f \in \operatorname{ext} S_{X^*} : f(x) = \|x\|\}$$

and

$$\mathcal{L}_Y = \{L \in \mathcal{L}(X, Y) : L_{|Y} = 0\}. \tag{3}$$

Theorem 2 (*See e.g. [13], [5, Prop. 2.1, p. 55]*)**.** *Let X be a finite-dimensional normed space. Then*

$$\operatorname{ext} S_{\mathcal{L}^*(X)} = \operatorname{ext} S_{X^*} \otimes \operatorname{ext} S_X,$$

where $(x^* \otimes x)(L) = x^*(Lx)$ for $x \in X$, $x^* \in X^*$ and $L \in \mathcal{L}(X, X)$.

Lemma 3 (*See e.g. [3]*)**.** *Assume that X is a normed space and let Y be a subspace of codimension k, $Y = \bigcap_{i=1}^{k} \ker f^i$, where the $f^i \in X^*$ are linearly independent. Then there exist $y^1, \ldots, y^k \in X$ satisfying $f^i(y^j) = \delta_{ij}$ for $i, j = 1, \ldots, k$ such that*

$$Px = x - \sum_{i=1}^{k} f^i(x) y^i \quad \text{for } x \in X.$$

Now we present a Kolmogorov type criterion which permits us to calculate the strong unicity constants of some subspaces of $\mathcal{L}(l_\infty^{(n)})$.

Theorem 4 (*See e.g. [14, Th. 2.1, p. 855]*)**.** *Let X be a normed space and let $Y \subset X$ be one of its subspaces. Assume that $x \in X \setminus Y$. Then $y_0 \in Y$ is the strongly unique best approximation to* *x (for short SUBA) with a constant $r > 0$ if and only if for any $y \in Y$ there exists $f \in E(x - y_0)$ such that* $\mathrm{Re} f(y) \leqslant -r\|y\|$.

2. Preliminary results

We start with a well-known lemma which characterizes the optimal strong uniqueness constants. This characterization first appeared in [2]. For the sake of completeness we will include the proof.

Lemma 5 (*See [2], [6, Th. 1.3, p. 84]*). *Let $V \subset X$, $\dim V = n$, $x_0 \in X \setminus V$, $v_0 \in V$. If v_0 is a SUBA to x_0 in V, then*

$$r = \inf_{v \in S_V} \left(\sup_{g \in E(x_0 - v_0)} g(v) \right)$$

is the strong unicity constant.

Proof. By Theorem 4, for every $v \in S_V$ there exists $g \in E(x_0 - v_0)$ such that $g(-v) < 0$. Consequently $g(v) > 0$ and $r \geqslant 0$. Denote

$$h(v) = \sup_{g \in E(x_0 - v_0)} g(v).$$

Let $\{v_n\}_{n=1}^{\infty} \subset S_V$ be such that

$$\lim_{n \to \infty} h(v_n) = \inf_{v \in S_V} h(v) = r. \tag{4}$$

By compactness of S_V, there exists a subsequence $\{n_k\}_{k=1}^{\infty}$ such that $\lim_{k \to \infty} v_{n_k} = \hat{v} \in S_V$. Notice that $h(\hat{v}) = r$. Suppose that this is not true, namely $h(\hat{v}) > r + \delta$ for some $\delta > 0$. Then $g(\hat{v}) > r + \delta$ for some $g \in E(x_0 - v_0)$. Since $\lim_{k \to \infty} g(v_{n_k}) = g(\hat{v})$,

$$h(v_{n_k}) \geqslant g(v_{n_k}) > r + \delta \quad \text{for } k \geqslant k_0,$$

which leads to a contradiction with (4).

Let $v \in V \setminus \{0\}$. Since $E(x_0 - v_0)$ is ω^*-closed, there exists $g \in E(x_0 - v_0)$ such that

$$g\left(-\frac{v}{\|v\|}\right) = h\left(-\frac{v}{\|v\|}\right) \geqslant r. \tag{5}$$

In particular $r = h(\hat{v}) = g(\hat{v})$ for some $g \in E(x_0 - v_0)$. Hence $r > 0$ and we can write

$$r = \min_{v \in S_V} \left(\max_{g \in E(x_0 - v_0)} g(v) \right).$$

By (5),

$$g(-v) \geqslant r\|v\| \quad \text{and} \quad g(v) \leqslant -r\|v\|,$$

which shows that the strong unicity constant is not less than r. Now we prove the opposite inequality. Let $r_1 > r$. Then there exists $v \in S_V$ such that

$$\sup_{g \in E(x_0 - v_0)} g(-v) < r_1.$$

Note that for every $g \in E(x_0 - v_0)$ we have $g(-v) < r_1$. This implies that $g(v) > -r_1\|v\|$ for every $g \in E(x_0 - v_0)$. By Theorem 4, r_1 is not the strong unicity constant. $\square$

Corollary 6. *Let* $0 < f_1 \leqslant f_2 \leqslant \cdots \leqslant f_k < 1/2$ *and* $\sum_{i=1}^k f_i = 1$. *Define*

$$\hat{U} := \left\{ y \in l_\infty^{(k)} : \sum_{i=1}^k f_i y_i = 0, \|y\|_\infty = \max_{i=1,\ldots,k} |y_i| = 1 \right\}. \tag{6}$$

Let us define

$$\hat{r} := \min_{y \in \hat{U}} \left(\max_{i=1,\ldots,k} (1 - 2f_i) y_i \right). \tag{7}$$

Then $\hat{r}$ *is the SUP-constant of a space* $\mathcal{P}(l_\infty^{(k)}, \hat{H})$, *where*

$$\hat{H} := \ker \hat{f}, \quad \hat{f} = (f_1, \ldots, f_k) \in \mathbb{R}^k.$$

Proof. Let $P_0 \in \mathcal{P}(l_\infty^{(k)}, \hat{H})$ be a minimal projection. By Lemma 3, P_0 has a form $P_0(x) = x - \hat{f}(x)y$, where $\hat{f}(y) = 1$. Since P_0 is a minimal projection, $0 \in \mathcal{L}_{\hat{H}}$ (see (3)) is the best approximation to P_0 in $\mathcal{L}_{\hat{H}}$. Assume that $L \in \mathcal{L}_{\hat{H}}$ and $\|L\| = 1$. Then $L(\cdot) = \hat{f}(\cdot)y$ for some $y \in \hat{H}$ such that $\|y\|_\infty = 1$. By Theorem 2 and [11, Ex. III.2.9, p. 103],

$$E(P_0) = \{e_i \otimes x^i : i = 1, \ldots, k\}, \tag{8}$$

where

$$x^i = (-1, \ldots, -1, 1_i, -1, \ldots, -1) \quad \text{for } i = 1, \ldots, k. \tag{9}$$

Hence we can write

$$\max_{g \in E(P_0)} g(L) = \max_{i=1,\ldots,k} (e_i \otimes x^i)(\hat{f}(\cdot)y) = \max_{i=1,\ldots,k} \hat{f}(x^i) y_i = \max_{i=1,\ldots,k} (2f_i - 1) y_i.$$

Consequently, by Lemma 5, the SUP-constant r is equal to

$$\begin{aligned} r &= \min_{\substack{L \in \mathcal{L}_{\hat{H}}, \\ \|L\|=1}} \left(\max_{g \in E(P_0)} g(L) \right) \\ &= \min_{\substack{y \in \hat{H}, \\ \|y\|_\infty = 1}} \left(\max_{i=1,\ldots,k} (2f_i - 1) y_i \right) = \min_{y \in \hat{U}} \left(\max_{i=1,\ldots,k} (1 - 2f_i) y_i \right). \quad \square \end{aligned}$$

Now we state a theorem which has been proved by V.V. Lokot in [8]. He showed that the SUP-constant of $P(l_\infty^{(k)}, \ker \hat{f})$, $0 < f_1 \leqslant f_2 \leqslant \cdots \leqslant f_k < 1/2$ is equal to

$$\frac{uf_1(1 - 2f_1)}{1 - 2f_1 - uf_1}, \quad \text{where } u = 1 - \|P_0\| = \left(\sum_{i=1}^k \frac{f_i}{1 - 2f_i} \right)^{-1}.$$

We also present a different proof of this result.

Theorem 7 (*See [8]*). *Let* $\hat{f} = (f_1, \ldots, f_k)$. $\hat{U}$ *and* $\hat{r}$ *are the same as in Corollary 6 (see (6), (7)). Then*

$$\hat{r} = \frac{f_1}{A}, \quad \textit{where } A = \sum_{i=2}^k \frac{f_i}{1 - 2f_i}. \tag{10}$$

Proof. Let $y \in \hat{U}$ yield the minimum in (7). By (6), there exists $i_0 \in \{1, \ldots, k\}$ such that $|y_{i_0}| = 1$. Now we consider two cases.

Case 1: $y_{i_0} = 1$.

If $i_0 = 1$, then $\frac{f_1}{1-2f_1} \leqslant \frac{f_i}{1-2f_i}$ (since $f_1 \leqslant f_i$) for all $i = 2, \ldots, k$. Hence

$$\frac{f_1}{(1-2f_1)} \leqslant \sum_{i=2}^{k} \frac{f_i}{1-2f_i},$$

and

$$(1-2f_1) \cdot A \geqslant f_1. \tag{11}$$

Suppose that $i_0 \in \{2, \ldots, k\}$. Then

$$(1-2f_{i_0}) \cdot A \geqslant (1-2f_k) \cdot \frac{f_k}{1-2f_k} = f_k \geqslant f_1. \tag{12}$$

By (11) and (12),

$$\max_{i=1,\ldots,k} (1-2f_i)y_i \geqslant (1-2f_{i_0}) \geqslant \frac{f_1}{A}. \tag{13}$$

Case 2: $y_{i_0} = -1$. Then

$$\sum_{\substack{i=1 \\ i \neq i_0}}^{k} f_i y_i = -f_{i_0} y_{i_0} = f_{i_0} \geqslant f_1.$$

Now we show that there exists $i_1 \in \{1, \ldots, k\} \setminus \{i_0\}$ such that $y_{i_1} \geqslant \frac{f_1}{(1-2f_{i_1}) \cdot A}$. Suppose that for every $i \in \{1, \ldots, k\} \setminus \{i_0\}$, $y_i < \frac{f_1}{(1-2f_i) \cdot A}$. Then

$$f_1 \leqslant f_{i_0} = \sum_{\substack{i=1 \\ i \neq i_0}}^{k} f_i y_i < \sum_{\substack{i=1 \\ i \neq i_0}}^{k} f_i \cdot \frac{f_1}{(1-2f_i) \cdot A} = \frac{f_1}{A} \left(A + \frac{f_1}{1-2f_1} - \frac{f_{i_0}}{1-2f_{i_0}} \right).$$

Hence

$$\frac{f_1}{1-2f_1} - \frac{f_{i_0}}{(1-2f_{i_0})} > 0,$$

which leads to a contradiction with $f_{i_0} \geqslant f_1$. Hence

$$\max_{i=1,\ldots,k} (1-2f_i)y_i \geqslant (1-2f_{i_1})y_{i_1} \geqslant \frac{f_1}{A}.$$

From both cases we get that $\hat{r} \geqslant \frac{f_1}{A}$.

For the converse inequality we find $y \in \hat{U}$ such that

$$\max_{i=1,\ldots,k} (1-2f_i)y_i = f_1/A.$$

Let us define

$$y = \left(-1, \frac{f_1}{(1-2f_2)A}, \frac{f_1}{(1-2f_3)A}, \ldots, \frac{f_1}{(1-2f_k)A} \right).$$

By (12) we obtain that $\|y\|_\infty = 1$. For every $i \in \{2, \dots, k\}$,

$$(1-2f_i)y_i = f_1/A \quad \text{and} \quad (1-2f_1)y_1 < 0.$$

Hence

$$\max_{i=1,\dots,k}(1-2f_i)y_i = f_1/A.$$

This implies that $\hat{r} \leqslant f_1/A$. The proof is completed. □

Example 8. (a) Let $f_1 = f_2 = \cdots = f_k = \frac{1}{k} < \frac{1}{2}$. Then

$$A = \sum_{i=2}^{k} \frac{1/k}{1-2/k} = \frac{k-1}{k-2} \quad \text{and} \quad \hat{r} = \frac{k-2}{k(k-1)}.$$

(b) Let $f_1 = f_2 = \cdots = f_{k-1} < f_k < \frac{1}{2}$. Then

$$A = (k-2)\frac{f_1}{1-2f_1} + \frac{1-(k-1)f_1}{2(k-1)f_1-1} = \frac{2f_1^2(k-1)^2 + f_1(1-2k)+1}{(1-2f_1)((2k-2)f_1-1)}$$

$$\text{and} \quad \hat{r} = \frac{f_1(1-2f_1)((2k-2)f_1-1)}{2f_1^2(k-1)^2 + f_1(1-2k)+1}.$$

3. The main result

Let us define

$$f^{(0)} = (f_1, f_2, \dots, f_k, 0, \dots, 0) \in \mathbb{R}^n, \tag{14}$$

$$f^{(j)} = (0, 0, \dots, 1_{j+k}, 0, \dots, 0) \in \mathbb{R}^n, \quad j = 1, 2, \dots, n-k. \tag{15}$$

To the end of this section we will assume that $k \geqslant 3$, $n > k$. Let

$$H = \bigcap_{j=0}^{n-k} \ker f^{(j)}. \tag{16}$$

Remark 9. Let $L \in \mathcal{L}_H$ (see (3)). Then L has the form

$$L(\cdot) = \sum_{j=0}^{n-k} f^{(j)}(\cdot) y^j,$$

where $y^j \in H$ for $j = 0, \dots, n-k$. Since $H \subset \bigcap_{j=0}^{n-k} \ker f^{(j)}$, we additionally get that $y_i^j = 0$ for $j = 0, \dots, n-k$, $i = k+1, \dots, n$ and $\sum_{i=1}^{k} f_i y_i^j = 0$ for $j = 0, \dots, n-k$.

Lemma 10. *Let $L \in \mathcal{L}_H$, $L(\cdot) = \sum_{j=0}^{n-k} f^{(j)}(\cdot) y^j$, where $y^j \in H$. Then*

$$\|L\| = \max_{i=1,\dots,k}\left(\sum_{j=0}^{n-k} |y_i^j|\right). \tag{17}$$

Proof.

$$\|L\| = \max_{\substack{i=1,\dots,n,\\ \|x\|_\infty=1}} |(Lx)_i| = \max_{i=1,\dots,n}\left(\max_{\|x\|_\infty=1}\left|\sum_{j=0}^{n-k} f^{(j)}(x) y_i^j\right|\right).$$

Since functionals $f^{(j)}$ have disjoint supports and $\|f^{(j)}\| = 1$ for $j = 0, \ldots, n-k$, we can select $x \in l_\infty^{(n)}$ such that

$$\|L\| = \max_{i=1,\ldots,n} \left(\sum_{j=0}^{n-k} |y_i^j| \right).$$

By Remark 9, $y_i^j = 0$ for $j = 0, \ldots, n-k,\ i = k+1, \ldots, n$, which implies (17). $\square$

Lemma 11. *Let us define*

$$U = \left\{ (y^0, \ldots, y^{n-k}) \in (l_\infty^{(n)})^{n-k+1} | y^j \in H,\ j = 0, \ldots, n-k, \right.$$
$$\left. \max_{i=1,\ldots,k} \sum_{j=0}^{n-k} |y_i^j| = 1 \right\}. \tag{18}$$

Then the SUP-constant of $\mathcal{P}(l_\infty^{(n)}, H)$ is equal to

$$r = \min \left\{ \max_{i=1,\ldots,k} \left((1 - 2f_i) y_i^0 + \sum_{j=1}^{n-k} |y_i^j| \right) :\ (y^0, \ldots, y^{n-k}) \in U \right\}. \tag{19}$$

Proof. Let $Q_0 \in \mathcal{P}(l_\infty^{(n)}, H)$ be a minimal projection. Then by (8)–(9) and [7, Th. 1.11]

$$E(Q_0) = \{e_i \otimes x, x \in A_i : i = 1, \ldots, k\}, \quad \text{where} \tag{20}$$
$$A_i = \{(-1, \ldots, -1, 1_i, -1, \ldots, -1_k, \pm 1, \ldots, \pm 1)\}, \quad i = 1, \ldots, k. \tag{21}$$

Since Q_0 is minimal projection, $0 \in \mathcal{L}_H$ is a best approximation to Q_0 in $\mathcal{L}_H$. Assume that $L \in \mathcal{L}_H$ and $\|L\| = 1$, i.e. $\max_{i=1,\ldots,k} \left(\sum_{j=0}^{n-k} |y_i^j| \right) = 1$ (see Lemma 10). By (20) and Remark 9, we can write

$$\begin{aligned}
\max_{g \in E(Q_0)} g(L) &= \max_{i=1,\ldots,k} \left(\max_{x \in A_i} (e_i \otimes x)(L(\cdot)) \right) \\
&= \max_{i=1,\ldots,k} \left(\max_{x \in A_i} \left(f^{(0)}(x) y_i^0 + \sum_{j=1}^{n-k} f^{(j)}(x) y_i^j \right) \right) \\
&= \max_{i=1,\ldots,k} \left((1 - 2f_i) y_i^0 + \max_{x \in A_i} \sum_{j=1}^{n-k} x_{j+k} y_i^j \right) \\
&= \max_{i=1,\ldots,k} \left((1 - 2f_i) y_i^0 + \sum_{j=1}^{n-k} |y_i^j| \right).
\end{aligned}$$

The last equality results from the definition of the sets A_i for $i = 1, \ldots, k$. Consequently, by Lemma 5, the SUP-constant r is equal to

$$r = \min_{\substack{L \in \mathcal{L}_H, \\ \|L\|=1}} \left(\max_{g \in E(Q_0)} g(L) \right)$$

$$= \min \left\{ \max_{i=1,\dots,k} \left((1-2f_i)y_i^0 + \sum_{j=1}^{n-k} |y_i^j| \right) : \ (y^0,\dots,y^{n-k}) \in U \right\}. \quad \square$$

Proposition 12. *Let $\hat{H} = \ker(f_1,\dots,f_k)$, $\sum_{i=1}^k f_i = 1$. We use the notation from (14)–(16). Let us denote by $\hat{r}$ and r the SUP-constant of $\mathcal{P}(l_\infty^{(k)}, \hat{H})$ and the SUP-constant of $\mathcal{P}(l_\infty^{(n)}, H)$, respectively. Then $r \leqslant \hat{r}$.*

Proof. Let $L \in \mathcal{L}_H$, $L = \sum_{j=0}^{n-k} f^{(j)} y^j$, where $(y^0,\dots,y^{n-k}) = (y^0,0,\dots,0) \in U$ and let $e_i \otimes x \in E(Q_0)$. Then

$$(e_i \otimes x)(L(\cdot)) = (L(x))_i = (f^{(0)}(x)y^0)_i = \hat{f}(x_1,\dots,x_k)y_i^0 = (e_i \otimes x^i)(\hat{f}(\cdot)y),$$

where $e_i \otimes x^i \in E(P_0)$ (see (8)–(9) and (20)–(21)), $y = (y_1^0,\dots,y_k^0) \in \hat{H}$ and $\hat{f}(\cdot)y \in S_{\mathcal{L}_{\hat{H}}}$. Hence

$$r = \inf_{L \in S_{\mathcal{L}_H}} \left(\sup_{g \in E(Q_0)} g(L) \right) \leqslant \inf_{L \in S_{\mathcal{L}_{\hat{H}}}} \left(\sup_{g \in E(P_0)} g(L) \right) = \hat{r}. \quad \square \tag{22}$$

Remark 13. For any $j = 0,1,\dots,n-k$ if $(y^0,\dots,y^{n-k}) \in U$ (see (18)), then $(y^0,\dots,y^{j-1},-y^j,y^{j+1},\dots,y^{n-k}) \in U$. Hence, by a form of the right-hand side of (19) we can assume that for some $i_0 \in \{1,\dots,k\} y_{i_0}^j \geqslant 0$ for all $j = 1,\dots,n-k$.

Now we prove the main result of this paper.

Theorem 14. *Let $\hat{f} = (f_1,\dots,f_k)$. $\hat{U}$ and $\hat{r}$ are the same as in Corollary 6. Consider $f^{(j)}$, $j = 0,\dots,n-k$ defined by (14)–(15) and H defined by (16). Then the SUP-constant $\hat{r}$ of a space $\mathcal{P}(l_\infty^{(k)}, \hat{H})$ is equal to the SUP-constant r of a space $\mathcal{P}(l_\infty^{(n)}, H)$.*

Proof. Assume to the contrary that $r < \hat{r}$ (compare with Proposition 12). Let $y \in U$ realize a minimum in (19). Take $i_0 \in \{1,\dots,k\}$ such that

$$\sum_{j=0}^{n-k} |y_{i_0}^j| = 1. \tag{23}$$

Applying Remark 13, we can assume that $y_{i_0}^j \geqslant 0$ for $j = 1,\dots,n-k$. Now consider two cases:

Case 1: $y_{i_0}^0 \geqslant 0$.
Note that

$$\sum_{j=1}^{n-k} y_{i_0}^j = 1 - y_{i_0}^0. \tag{24}$$

By our hypothesis

$$(1-2f_{i_0})y_{i_0}^0 + \sum_{j=1}^{n-k} |y_{i_0}^j| < \hat{r}. \tag{25}$$

By (24) this is equivalent to

$$(1-2f_{i_0})y_{i_0}^0 + 1 - y_{i_0}^0 < \hat{r},$$

which gives

$$y_{i_0}^0 > \frac{1-\hat{r}}{2f_{i_0}}. \tag{26}$$

By (13),

$$1 - 2f_{i_0} \geqslant \frac{f_1}{A} = \hat{r},$$

which combined with (26) leads to

$$y_{i_0}^0 > \frac{1-\hat{r}}{2f_{i_0}} \geqslant 1,$$

a contradiction with (23).

Case 2: $y_{i_0}^0 < 0$.

We have

$$\sum_{j=1}^{n-k} y_{i_0}^j = 1 + y_{i_0}^0. \tag{27}$$

By our hypothesis

$$(1-2f_i)y_i^0 + \sum_{j=1}^{n-k} |y_i^j| < \hat{r} \quad \text{for } i = 1, \ldots, k.$$

Multiplying each of the above inequalities by $\frac{f_i}{1-2f_i}$ for $i = 1, \ldots, k$ we get

$$\sum_{j=1}^{n-k} \frac{f_i}{1-2f_i}|y_i^j| < \frac{f_i}{1-2f_i}\hat{r} - f_i y_i^0 \quad \text{for } i = 1, \ldots, k. \tag{28}$$

Summing up inequalities (28) for $i = 1, \ldots, k,\ i \neq i_0$, we conclude that

$$\sum_{\substack{i=1\\ i\neq i_0}}^{k} \sum_{j=1}^{n-k} \frac{f_i}{1-2f_i}|y_i^j| < \sum_{\substack{i=1\\ i\neq i_0}}^{k} \frac{f_i}{1-2f_i}\hat{r} - \sum_{\substack{i=1\\ i\neq i_0}}^{k} f_i y_i^0. \tag{29}$$

Since $y^j \in \ker\ f^{(0)}$ for $j = 0, \ldots, n-k$,

$$\sum_{\substack{i=1\\ i\neq i_0}}^{k} f_i y_i^j = -f_{i_0} y_{i_0}^j \quad \text{for } j = 0, 1, \ldots, n-k. \tag{30}$$

Combining (29) with (30) we get

$$\sum_{\substack{i=1\\ i\neq i_0}}^{k} \sum_{j=1}^{n-k} \frac{f_i}{1-2f_i}|y_i^j| < \left(A + \frac{f_1}{1-2f_1} - \frac{f_{i_0}}{1-2f_{i_0}}\right)\hat{r} + f_{i_0} y_{i_0}^0. \tag{31}$$

Since $f_1 \leqslant f_{i_0}$,

$$-\frac{1}{1-2f_1}\sum_{j=1}^{n-k}\sum_{\substack{i=1\\ i\neq i_0}}^{k} f_i y_i^j \leqslant \sum_{\substack{i=1\\ i\neq i_0}}^{k}\sum_{j=1}^{n-k} \frac{f_i}{1-2f_1}|y_i^j| \leqslant \sum_{\substack{i=1\\ i\neq i_0}}^{k}\sum_{j=1}^{n-k} \frac{f_i}{1-2f_i}|y_i^j|. \tag{32}$$

By (30)–(32) we obtain

$$\frac{1}{1-2f_1}\sum_{j=1}^{n-k} f_{i_0} y_{i_0}^j < \left(A + \frac{f_1}{1-2f_1} - \frac{f_{i_0}}{1-2f_{i_0}}\right)\hat{r} + f_{i_0} y_{i_0}^0.$$

Applying (27), taking into account that $A \cdot \hat{r} = f_1$ (see (10)) and $f_1 \leqslant f_{i_0}$, we get

$$\frac{f_{i_0}}{1-2f_1}(1+y_{i_0}^0) < \left(\frac{f_1}{1-2f_1} - \frac{f_{i_0}}{1-2f_{i_0}}\right)\hat{r} + f_{i_0}(1+y_{i_0}^0). \tag{33}$$

Note that (33) is equivalent to

$$y_{i_0}^0 < \frac{1-2f_1}{2f_1 f_{i_0}}\left(\frac{f_1}{1-2f_1} - \frac{f_{i_0}}{1-2f_{i_0}}\right)\hat{r} - 1. \tag{34}$$

Since $f_1 \leqslant f_{i_0}$, inequality (34) implies $y_{i_0}^0 < -1$, a contradiction with (23). □

Remark 15. The SUP-constant of $\mathcal{P}(l_\infty^{(4)}, H)$, where $H = \ker f \cap \ker g$, $f = (1, r, s, 0)$, $g = (0, 0, 0, 1)$, was found by Martinov in [9]. Theorem 14 solves the conjecture stated in [12, Rem. 4, p. 120] and generalizes [9].

Now we shall investigate the case when a norm of minimal projection is equal to 1. First we prove a preliminary result.

Lemma 16. *Let $0 \leqslant f_1 \leqslant \cdots \leqslant f_{k-1} < f_k$, $f_k \geqslant 1/2$ and $f_{k-1} < 1/2$. Consider $f^{(j)}$, $j = 0, \ldots, n-k$ defined by (14)–(15) and H defined by (16). Assume that $Q \in \mathcal{P}(l_\infty^{(n)}, H)$ and let $Q_0 \in \mathcal{P}(l_\infty^{(n)}, H)$ be a minimal projection. Then*

$$\|Q - Q_0\| = \max_{i=1,\ldots,k-1} \sum_{j=0}^{n-k} |y_i^j|.$$

Proof. Since $y^j \in H$ for $j = 0, \ldots, n-k$,

$$-f_k y_k^j = \sum_{i=1}^{k-1} f_i y_i^j, \quad j = 0, \ldots, n-k.$$

Hence

$$f_k |y_k^j| = \left|\sum_{i=1}^{k-1} f_i y_i^j\right|, \quad j = 0, \ldots, n-k. \tag{35}$$

Summing up Eq. (35) for $j = 0, \ldots, n-k$ we get

$$\begin{aligned} f_k \sum_{j=0}^{n-k} |y_k^j| &= \sum_{j=0}^{n-k}\left|\sum_{i=1}^{k-1} f_i y_i^j\right| \leqslant \sum_{j=0}^{n-k}\sum_{i=1}^{k-1} f_i |y_i^j| \leqslant \sum_{i=1}^{k-1} f_i \max_{i=1,\ldots,k-1}\sum_{j=0}^{n-k}|y_i^j| \\ &= (1-f_k)\max_{i=1,\ldots,k-1}\sum_{j=0}^{n-k}|y_i^j| \leqslant f_k \max_{i=1,\ldots,k-1}\sum_{j=0}^{n-k}|y_i^j|. \end{aligned} \tag{36}$$

In the last inequality we have applied the fact that $f_k \geqslant \frac{1}{2}$. Dividing (36) by f_k we get

$$\sum_{j=0}^{n-k} |y_k^j| \leqslant \max_{i=1,\ldots,k-1} \sum_{j=0}^{n-k} |y_i^j|.$$

Applying Lemma 10 we get the desired equality. $\square$

Theorem 17. *Let* $0 \leqslant f_1 \leqslant \cdots \leqslant f_k$*, where* $f_k \geqslant 1/2$, $f_{k-1} < 1/2$, $k \geqslant 3$ *and* $n > k$*. Consider* $\hat{f} = \ker(f_1, \ldots, f_k)$*,* $f^{(j)}$, $j = 0, \ldots, n-k$ *defined by* (14)–(15) *and* H *defined by* (16)*. Then the SUP-constant of a space* $\mathcal{P}(l_\infty^{(k)}, \hat{H})$ *is equal to the SUP-constant of a space* $\mathcal{P}(l_\infty^{(n)}, H)$*.*

Proof. Let $P_0 \in \mathcal{P}(l_\infty^{(k)}, \hat{H})$ and $Q_0 \in \mathcal{P}(l_\infty^{(n)}, H)$ be the minimal projections. Then (see [7]) $\|P_0\| = \|Q_0\| = 1$ and

$$P_0(x) = x - \hat{f}(x)y, \quad x \in l_\infty^{(k)}, \quad y = (0, \ldots, 0, 1/f_k), \tag{37}$$
$$Q_0(x) = (P_0(x_1, \ldots, x_k), 0, \ldots, 0), \quad x \in l_\infty^{(n)}.$$

Let $Q \in \mathcal{P}(l_\infty^{(n)}, H)$. Notice that $Q - Q_0 \in \mathcal{L}_H$ and $Q(\cdot) - Q_0(\cdot) = \sum_{j=0}^{n-k} f^{(j)}(\cdot)y^j$, where $y^j \in H$ for $j = 0, \ldots, n-k$ (see Remark 9). By Lemma 16, for some $i_0 \in \{1, \ldots, k-1\}$,

$$\|Q - Q_0\| = \sum_{j=0}^{n-k} |y_{i_0}^j|.$$

Since $y_{i_0} = 0$, we get that $(e_{i_0} \circ P_0)(x) = x_{i_0}$ (see (37)). Consequently,

$$\begin{aligned}
\|Q\| \geqslant \|e_{i_0} \circ Q\| &= \max_{\|x\|_\infty = 1} \left| (e_{i_0} \circ P_0)(x) + \sum_{j=0}^{n-k} f^{(j)}(x) y_{i_0}^j \right| \\
&= \max_{\|x\|_\infty = 1} \left| x_{i_0} + \sum_{i=1}^{k} f_i x_i y_{i_0}^0 + \sum_{j=1}^{n-k} x_{k+j} y_{i_0}^j \right| \\
&= \max_{\|x\|_\infty = 1} \left| x_{i_0}(1 + f_{i_0} y_{i_0}^0) + \sum_{\substack{i=1 \\ i \neq i_0}}^{k} f_i x_i y_{i_0}^0 + \sum_{j=1}^{n-k} x_{k+j} y_{i_0}^j \right| \\
&= |1 + f_{i_0} y_{i_0}^0| + (1 - f_{i_0}) \left| y_{i_0}^0 \right| + \sum_{j=1}^{n-k} \left| y_{i_0}^j \right| \\
&\geqslant 1 + (1 - 2f_{i_0}) \sum_{j=0}^{n-k} |y_{i_0}^j| \geqslant \|Q_0\| + (1 - 2f_{k-1}) \|Q - Q_0\|.
\end{aligned}$$

The last inequality results from $f_{i_0} \leqslant f_{k-1}$ for $i_0 \in \{1, \ldots, k-1\}$. It is well known that the SUP-constant of a space $\mathcal{P}(l_\infty^{(k)}, \hat{H})$ is equal to $\hat{r} = 1 - 2f_{k-1}$ [11, Th. III.3.1, p. 105]. Hence we have shown that $r \geqslant 1 - 2f_{k-1} = \hat{r}$. By Proposition 12, the proof is complete. $\square$

Remark 18 (*See [11, Theorem III.3.1, p. 105]*)**.** Let $P_0 \in \mathcal{P}(l_\infty^{(k)}, \ker \hat{f})$ be the minimal projection, where $\hat{f} = (f_1, \ldots, f_k)$, $\sum_{i=1}^{k} |f_i| = 1$.

If $\|P_0\| = 1$, then P_0 is the SUM-projection if and only if $|f_i| \geqslant 1/2$ for exactly one index $i \in \{1, \ldots, k\}$.

If $\|P_0\| > 1$, then P_0 is the SUM-projection if and only if $0 < |f_i| < 1/2$ for $i = 1, \ldots, k$.

For results involving SUM-projections onto hyperplanes in l_∞^n see also, for example, [1].

Remark 19 (*See [11, Th. III.3.1, p. 105]*)**.** Let $f = (f_1, \ldots, f_k) \in \mathbb{R}^k$ and $\sum_{i=1}^k |f_i| = 1$. Let σ be a permutation of the set $\{1, \ldots, k\}$. Set $\hat{f} = (|f_{\sigma_1}|, \ldots, |f_{\sigma_k}|)$. Then the SUP-constant of $\mathcal{P}(l_\infty^{(k)}, \ker \hat{f})$ is equal to the SUP-constant of $\mathcal{P}(l_\infty^{(k)}, \ker f)$.

Proof. Let $P \in \mathcal{P}(l_\infty^{(k)}, \ker f)$, $P(x) = x - f(x)y$. Set $\bar{y}_i = y_{\sigma_i}$ if $f_{\sigma_i} = 0$, $\bar{y}_i = f_{\sigma_i}/|f_{\sigma_i}|y_{\sigma_i}$ if $f_{\sigma_i} \neq 0$. Let $\hat{P}(x) = x - \hat{f}(x)\bar{y}$. Then $\hat{P} \in \mathcal{P}(l_\infty^{(k)}, \ker \hat{f})$ and $\|\hat{P}\| = \|P\|$. Hence the mapping $P \to \hat{P}$ is an isometry from $\mathcal{P}(l_\infty^{(k)}, \ker f)$ onto $\mathcal{P}(l_\infty^{(k)}, \ker \hat{f})$. This completes the proof. $\square$

Now we can state the main result of this paper in a more general form.

Theorem 20. *Consider $f^{(j)}$, $j = 0, \ldots, n-k$, defined by (14)–(15) and H defined by (16). Let $\hat{f} = (f_1, \ldots, f_k) \in \mathbb{R}^k$, $\sum_{i=1}^k |f_i| = 1$ and $\hat{H} = \ker \hat{f}$. Assume that the minimal projection $P_0 \in \mathcal{P}(l_\infty^{(k)}, \ker \hat{f})$ is the SUM-projection. Then the minimal projection $Q_0 \in \mathcal{P}(l_\infty^{(n)}, H)$ is the SUM-projection and the SUP-constant $\hat{r}$ of a space $\mathcal{P}(l_\infty^{(k)}, \hat{H})$ is equal to the SUP-constant r of a space $\mathcal{P}(l_\infty^{(n)}, H)$. Additionally:*

If $\|P_0\| = 1$, then there exists exactly one index $i \in \{1, \ldots, k\}$ such that $|f_i| \geqslant 1/2$ and $r = \hat{r} = \min\{1 - 2|f_j| : j = 1, \ldots, k, j \neq i\}$.

If $\|P_0\| > 1$, then $0 < |f_i| < 1/2$ for $i = 1, \ldots, k$ and

$$r = \hat{r} = \frac{u|f_{i_0}|(1 - 2|f_{i_0}|)}{1 - 2|f_{i_0}| - u|f_{i_0}|},$$

where $u = 1 - \|P_0\| = \left(\sum_{i=1}^k \frac{|f_i|}{1-2|f_i|}\right)^{-1}$ and $|f_{i_0}| = \min\{|f_j| : j = 1, \ldots, k\}$.

References

[1] M. Baronti, G. Lewicki, Strongly unique minimal projections on hyperplanes, J. Approx. Theory 78 (1994) 1–18.
[2] M.W. Bartelt, H.W. McLaughlin, Characterizations of strong unicity in approximation theory, J. Approx. Theory 9 (1973) 255–266.
[3] J. Blatter, E.W. Cheney, Minimal projections onto hyperplanes in sequence spaces, Ann. Mat. Pura Appl. 101 (1974) 215–227.
[4] E.W. Cheney, Introduction to Approximation Theory, AMS Chelsea Publishing, 2000.
[5] H.S. Collins, W. Ruess, Weak compactness in spaces of compact operators and of vector-valued functions, Pacific J. Math. 106 (1) (1983) 45–71.
[6] G. Lewicki, Strong unicity criterion in some space of operators, Comment. Math. Univ. Carolin. 34 (1) (1993) 81–87.
[7] G. Lewicki, On minimal projections in $l_\infty^{(n)}$, Monatsh. Math. 129 (2000) 119–131.
[8] V.V. Lokot, The constants of strongly unique minimal projections onto hyperplanes in $l_\infty^n (n \geq 3)$, Mat. Zametki 72 (5) (2002) 723–728.
[9] O.M. Martinov, Constants of strong unicity of minimal projections onto some two-dimensional subspaces of $l_\infty^{(4)}$, J. Approx. Theory 118 (2002) 175–187.
[10] D.J. Newman, H.S. Shapiro, Some theorems on Chebyshev approximation, Duke Math. J. 30 (4) (1963) 673–681.
[11] W. Odyniec, G. Lewicki, Minimal Projections in Banach Spaces, in: Lecture Notes in Mathematics, vol. 1449, Springer, Berlin, Heidelberg, New York, 1990.
[12] W. Odyniec, M.P. Prophet, A lower bound of the strongly unique minimal projection constant of $l_\infty^n, n \geqslant 3$, J. Approx. Theory 145 (2007) 111–121.
[13] W. Ruess, C. Stegall, Extreme points in duals of operator spaces, Math. Ann. 261 (1982) 535–546.
[14] A. Wójcik, Characterization of strong unicity by tangent cones, in: Z. Ciesielski (Ed.), Approximation and Function Spaces, Proceedings of the International Conference, Gdańsk, 1979, PWN, North-Holland, Amsterdam, Warsaw, 1981, pp. 854–866.